

ON DIAGONAL ENTRIES OF CARTAN MATRICES OF p -BLOCKS

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ABSTRACT. In this short note, we show some inequalities on Cartan matrices, centers and socles of blocks of group algebras. Our main theorems are generalizations of the facts on dimensions of Reynolds ideals.

1. INTRODUCTION

Let p be a prime, G a finite group and (K, \mathcal{O}, F) a splitting p -modular system for G where \mathcal{O} is a complete discrete valuation ring with quotient field K of characteristic 0 and residue field F of characteristic p . For each block B of the group algebra FG , we denote by $k(B)$ and $l(B)$ the numbers of irreducible ordinary and Brauer characters associated to B , respectively. Our purpose of this note is to show some inequalities on the Cartan matrix C_B , the center $Z(B)$ and the n -th socle $\text{soc}^n(B)$ of B . In this note, for any integer $n \geq 1$, $c(B, n)$ denotes the sum of multiplicities of S as composition factors in the factor module $P_S/P_S J^n$ where S ranges over isomorphism classes of irreducible right B -modules, P_S is the projective cover of S and J is the Jacobson radical of B . Therefore, for example, $c(B, 1) = l(B)$, $c(B, 2) = l(B) + \sum_S \dim \text{Ext}_B^1(S, S)$ and $c(B, \lambda) = \text{tr } C_B$ for the Loewy length λ of B . In this note, we prove the following theorems.

Theorem 1.1. *For any $1 \leq n \leq \lambda$,*

$$\dim \text{soc}^n(B) \cap Z(B) \leq c(B, n).$$

Theorem 1.2. *Assume $2 \leq \lambda$. Then there exists an integer $2 \leq m \leq \lambda$ such that*

$$\dim \text{soc}^n(B) \cap Z(B) = c(B, n),$$

$$\dim \text{soc}^{n'}(B) \cap Z(B) \leq c(B, n')$$

for all $1 \leq n \leq m < n' \leq \lambda$.

We mention some previous results as remarks of the main theorems. It is well known that $\dim \text{soc}(B) \cap Z(B) = l(B)$ and Okuyama has shown that $\dim \text{soc}^2(B) \cap Z(B) = l(B) + \sum_S \dim \text{Ext}_B^1(S, S)$ in [4] (the article is written in Japanese, so see [3] for the original proof by Okuyama or see Proposition 3.4 in this paper). Therefore Theorem 1.2 is a generalization of these facts. Moreover we can obtain the famous inequality $k(B) \leq \text{tr } C_B$ as a corollary to Theorem 1.1 as will be proved later.

2. PRELIMINARIES

This section is devoted to some notations and fundamental properties of finite-dimensional symmetric algebra A over F with a bilinear form $\langle \cdot, \cdot \rangle : A \times A \rightarrow F$. The facts described in this section are applied to the basic algebra of B .

For a subspace U of A , we define

$$\begin{aligned}\text{Ann}_A(U) &= \{a \in A \mid Ua = 0\}, \\ U^\perp &= \{a \in A \mid \langle U, a \rangle = 0\}.\end{aligned}$$

Lemma 2.1. *Let U, V be two subspaces of A . Then the following hold:*

- (1) $(U^\perp)^\perp = U$.
 $(U + V)^\perp = U^\perp \cap V^\perp$.
 $(U \cap V)^\perp = U^\perp + V^\perp$.
- (2) If $V \subseteq U$, then $U^\perp \subseteq V^\perp$.
- (3) $\dim U^\perp = \dim A - \dim U$.
- (4) If U is an ideal of A , then $\text{Ann}_A(U) = U^\perp$.

Furthermore, we define the *commutator subspace* of subspaces U and V of A by

$$[U, V] = \sum_{u \in U, v \in V} F(uv - vu).$$

By the definition above, the next lemma is clear.

Lemma 2.2. *Let U, V and W be subspaces of A . Then we have*

$$\begin{aligned}[U + V, W] &= [U, W] + [V, W], \\ [U, V + W] &= [U, V] + [U, W].\end{aligned}$$

In particular, the next lemma is important.

Lemma 2.3 ([2, Lemma A]). $[A, A]^\perp = Z(A)$.

Now let $e_1, \dots, e_{l(B)}$ be representatives for the conjugacy classes of primitive idempotents in B . Thus $c_{ij} = \dim e_i B e_j$ and $c(B, n) = \sum_{1 \leq i \leq l(B)} \dim e_i B e_i / e_i J^n e_i$ where $C_B = (c_{ij})$. We put $e = e_1 + \dots + e_{l(B)}$ and denote by eBe the basic algebra of B . Then B and eBe are symmetric algebras over F . Moreover they are Morita equivalent since $B = BeB$, and hence the next lemma holds.

Lemma 2.4. *For an ideal I of B , eIe is that of eBe and we have*

$$\dim \text{Ann}_B(I) \cap Z(B) = \dim \text{Ann}_{eBe}(eIe) \cap Z(eBe).$$

Finally, we define a subspace

$$B(n) = \sum_{1 \leq i \leq l(B)} e_i J^n e_i + \sum_{1 \leq i \neq j \leq l(B)} e_i B e_j$$

of eBe for each $n \geq 1$. Since $eBe = \sum_{1 \leq i, j \leq l(B)} e_i B e_j$ and $B(n)$ are direct sums, we deduce the next lemma by Lemma 2.1.

Lemma 2.5.

$$\begin{aligned}\dim eBe &= \sum_{1 \leq i, j \leq l(B)} c_{ij}, \\ \dim B(n)^\perp &= c(B, n).\end{aligned}$$

3. PROOF OF MAIN THEOREMS

Theorem 1.1 is due to the next lemma.

Lemma 3.1. *If $i \neq j$, then $e_i Be_j \subseteq [eBe, eBe]$.*

Proof. For any $x \in e_i Be_j$, we can write $x = xe_j - e_j x$. So $x \in [e_i Be_j, e_j Be_j] \subseteq [eBe, eBe]$. \square

Now we prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 3.1, $B(n) \subseteq eJ^n e + [eBe, eBe]$ and hence $\text{Ann}_{eBe}(eJ^n e) \cap Z(eBe) \subseteq B(n)^\perp$ using Lemma 2.1 and 2.3. So Lemma 2.4 and 2.5 gives us that $\dim \text{soc}^n(B) \cap Z(B) = \dim \text{Ann}_B(J^n) \cap Z(B) \leq c(B, n)$ as claimed. \square

We show a corollary to Theorem 1.1. We substitute λ for n and thus we obtain the next inequality (remark $k(B) = \dim Z(B)$ and see also [1, Proposition 4.2] or [5, Theorem A]).

Corollary 3.2. $k(B) \leq \text{tr } C_B$.

To prove Theorem 1.2, we have to see the structure of $[eBe, eBe]$ as follows.

Lemma 3.3. $[eBe, eBe] \subseteq \sum_{1 \leq i, j \leq l(B)} [e_i Be_j, e_j Be_i] + \sum_{1 \leq i \neq j \leq l(B)} e_i Be_j$.

Proof. We first obtain $[eBe, eBe] = \sum_{1 \leq i, j, s, t \leq l(B)} [e_i Be_j, e_s Be_t]$ by Lemma 2.2. Thereby we investigate three cases in the following.

Case 1 $i \neq t$ and $j \neq s$.

Since $e_t e_i = e_j e_s = 0$, $[e_i Be_j, e_s Be_t] = 0$.

Case 2 $i = t$ and $j \neq s$.

Clearly, we have $[e_i Be_j, e_s Be_i] \subseteq e_s Be_i Be_j \subseteq e_s Be_j$ where $j \neq s$.

Case 3 $i \neq t$ and $j = s$.

By similar way above, $[e_i Be_j, e_j Be_t] \subseteq e_i Be_t$ where $i \neq t$.

Thus the claim follows since the remaining case is that $i = t$ and $j = s$. \square

Theorem 1.2 is a direct consequence of the next proposition.

Proposition 3.4. *The following are equivalent:*

- (1) $\dim \text{soc}^n(B) \cap Z(B) = c(B, n)$.
- (2) $[e_i Be_j, e_j Be_i] \subseteq e_i J^n e_i + e_j J^n e_j$ for all $1 \leq i, j \leq l(B)$.

Proof. By the proof of Theorem 1.1, (1) holds if and only if $eJ^n e + [eBe, eBe] \subseteq B(n)$. However, by Lemma 3.1 and 3.3, we have

$$\begin{aligned} eJ^n e + [eBe, eBe] &= \sum_{1 \leq i \leq l(B)} e_i J^n e_i + [eBe, eBe] \\ &= B(n) + \sum_{1 \leq i, j \leq l(B)} [e_i Be_j, e_j Be_i]. \end{aligned}$$

So it is clear that (2) implies (1). On the other hand, suppose (1) holds and $x \in [e_i Be_j, e_j Be_i]$. Thus $x \in [eBe, eBe] \subseteq B(n)$. Since we can write $x = e_i x e_i + e_j x e_j$, we deduce that $x \in e_i J^n e_i + e_j J^n e_j$, as required. \square

It is easy to show the result of Okuyama in [4] by using this proposition. If $i \neq j$, then $e_i B e_j = e_i J e_j$ and so $[e_i B e_j, e_j B e_i] = [e_i J e_j, e_j J e_i] \subseteq e_i J^2 e_i + e_j J^2 e_j$. On the other hand, $[e_i B e_i, e_i B e_i] = [F e_i + e_i J e_i, F e_i + e_i J e_i] \subseteq e_i J^2 e_i$ since $e_i B e_i$ is local. Therefore $n = 2$ satisfies the conditions above.

Now we prove Theorem 1.2. We fix the largest integer $2 \leq m \leq \lambda$ satisfies the conditions in Proposition 3.4. Then any integer n (such that $n \leq m$) holds same properties since $J^m \subseteq J^n$. Therefore we have completed the proof of the theorem.

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